Design of edge detector

General method.

1. Express class of features in mathematical form.

   (a) intensity function:

   $$f(x, p_1, p_2, \ldots, p_k)$$

   where x is coordinate

   p_i are parameters

   (b) ~~constraints~~ constraint on parameters:

   $$C(p_1, \ldots, p_n)$$

   where C is a computable predicate.

2. "Digitize" the intensity function f. Digitization is assumed to average over each pixel, and to add noise:

   $$a_i = f_i + n_i$$

   where $f_i = \int_i^{i+1} f(x)\,dx$

3. We wish to test whether a set of samples a_i from the image is a feature. If we were testing a continuous function, we would have two tests:

   (a) can the function be expressed in the form f?

   (b) if so, do the parameters satisfy the constraint C?

Since we are working with samples with added noise, we can always satisfy (a) by assuming arbitrarily large values for the noise. So instead of (a) we use:

(a') if we assume that the samples a_i represent a feature, can we find parameters for the feature such that the added noise is "acceptable"?

The total root mean square noise N is given by:

$$N^2 = \sum_i n_i^2 = \sum_i (a_i - f_i)^2$$

We want to find values for the parameters p_i such that N is minimized. At a minimum the partial derivatives of N w.r.t. p_i will be zero:

$$\frac{\partial N}{\partial p_i} = 0$$

$$\therefore \quad \frac{\partial}{\partial p_i} \sum_k (a_k - f_k)^2 = 0$$

$$\therefore \quad \sum_k (a_k - f_k) \frac{\partial f_k}{\partial p_i} = 0$$

This gives us a set of simultaneous equations, which we solve, yielding expressions for the parameters p_i in terms of the samples a_i.

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4. By substituting the expressions just obtained for the f_i into the expression for the constraint c, we can express c in terms of the samples a_i. In this form c can be directly computed from the samples, and so can be used as part of the feature detector.

5. Finally, we have to devise a test for whether the noise is "acceptable". The noise assumed to be added to each sample is given by $n_i = a_i - f_i$, and this can expressed in terms of the samples alone by substituting the ~~value~~ expressions found for the parameters in step 3.
   If the average noise level for the image is known, we could simply require that each n_i is less than a fixed threshold.
   Alternatively, note that we are normally concerned with several classes of features (and non-features, e.g. uniform intensity). We can then require that the noise be sufficiently small that no set of sample points is accepted into more than one class. The edge detector uses an ad hoc version of this approach.

Edge detector.

1. The intensity function is:

$$f(x) = \begin{cases} l & \text{if } x < d \\ r & \text{if } x \geq d \end{cases}$$

where the parameters are:

l	intensity to left of edge
r	intensity to right of edge
d	distance of edge from origin.

The origin is at the left of a window of width 5.
The constraints are:

C1. $$|l - r| > t$$
where t is a threshold. This is used to distinguish edges from regions of constant intensity, which can also be expressed in ~~the~~ form given for $f(x)$ above.

C2. $d > 2$
C3. $d \leq 3$
These require that the edge pass through the centre pixel of the window.

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2.
$$f_0 = l$$
$$f_1 = l$$
$$f_2 = (d-2)l + (3-d)r$$
$$f_3 = r$$
$$f_4 = r$$

3.
$$\sum_k (a_k - f_k)\frac{\partial f_k}{\partial l} = 0$$

yields

$$\cancel{(a_0 - 2f_0)2 + (a_1 - 2f}$$
$$(a_0 - l) + (a_1 - l) + [a_2 - (d-2)l - (3-d)r](d-2) = 0$$

$$\sum_k (a_k - f_k)\frac{\partial f_k}{\partial r} = 0$$

yields

$$[a_2 - (d-2)l - (3-d)r](3-d) + (a_3 - r) + (a_4 - r) = 0$$

$$\sum_k (a_k - f_k)\frac{\partial f_k}{\partial d} = 0$$

yields

$$[a_2 - (d-2)l - (3-d)r](r-l) = 0$$

Assuming $l \neq r$, the solution of these equations is:

$$l = \tfrac{1}{2}(a_0 + a_1)$$
$$r = \tfrac{1}{2}(a_3 + a_4)$$
$$d = \frac{-2a_0 - 2a_1 - 2a_2 + 3a_3 + 3a_4}{-a_0 - a_1 + a_3 + a_4}$$

4. substituting for the parameters in C1:

$$\tfrac{1}{2}\left|(a_0 + a_1) - (a_3 + a_4)\right| > t$$

substituting for parameters in C2:

$$\frac{-2a_0 - 2a_1 - 2a_2 + 3a_3 + 3a_4}{-a_0 - a_1 + a_3 + a_4} > 2$$

which simplifies to:

if $(a_3 + a_4) > (a_0 + a_1)$
then $a_2 < \tfrac{1}{2}(a_3 + a_4)$
else $a_2 > \tfrac{1}{2}(a_3 + a_4)$

similarly, C3 yields:

if $(a_3 + a_4) > (a_0 + a_1)$
then $a_2 \geq \tfrac{1}{2}(a_0 + a_1)$
else $a_2 \leq \tfrac{1}{2}(a_0 + a_1)$

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5. $$n_0 = a_0 - l = \tfrac{1}{2}(a_0 - a_1)$$
$$n_1 = a_1 - l = \tfrac{1}{2}(a_1 - a_0)$$
$$n_2 = a_2 - (d-2)l - (3-d)r = 0$$
$$n_3 = a_3 - r = \tfrac{1}{2}(a_3 - a_4)$$
$$n_4 = a_4 - r = \tfrac{1}{2}(a_4 - a_3)$$

The criterion for noise acceptability is $|n_i| < n$, for each i, where n is threshold determined below. Since $|n_0| = |n_1|$, $|n_3| = |n_4|$, and $n_2 = 0$, there are only two tests:

$$\tfrac{1}{2}|a_0 - a_1| < n$$
$$\tfrac{1}{2}|a_3 - a_4| < n$$

The threshold n is chosen by considering just one class of non-feature: uniform gradient. Suppose we have an edge passing through the centre of the window, with $l < r$. The worst case (from the point of view of making the edge look like a gradient) assignment of noise is:

$$a_0 = l - n_W$$
$$a_1 = l + n_W$$
$$a_2 = \tfrac{1}{2}(l + r)$$
$$a_3 = r - n_W$$
$$a_4 = r + n_W$$

For this to be a uniform gradient we must have

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3$$

i.e. $$2n_W = \tfrac{1}{2}(r-l) - n_W = \tfrac{1}{2}(r-l) - n = 2n_W$$

i.e. $$n_W = \tfrac{1}{6}(r-l)$$

If we use equal noise margins for both edges and gradients, the noise threshold n should be $\frac{1}{2}|n_W|$:

$$n = \tfrac{1}{2}|n_W|$$
$$= \tfrac{1}{12}|r-l|$$
$$= \tfrac{1}{24}|a_3 + a_4 - a_0 - a_1|$$

So the noise tests are:

$$|a_0 - a_1| < \tfrac{1}{12}|a_3 + a_4 - a_0 - a_1|$$
$$|a_3 - a_4| < \tfrac{1}{12}|a_3 + a_4 - a_0 - a_1|$$

Design of bar detector.

This follows the same general method used for the edge detector.

1. The intensity function is:

$$f(x) = \begin{cases} m & \text{if } l < x \leq r \\ s & \text{otherwise} \end{cases}$$

where the parameters are:

m	intensity in middle of bar
s	intensity at each side of bar
l	distance of left edge of bar from origin
r	" " right " " " " "

The constraints are:

C1. $l > 1$

C2. $l \leq 2$

C3. $r > 3$

C4. $r \leq 4$

These constraints require the width of the bar to be less than 3, since wider bars will be detected as pairs of edges.
They also require that neither edge of the bar be in the centre pixel of the window. Such bars are indistinguishable from the ones considered here, due to digitization.

~~C4.~~ They also require that centre of bar is in middle pixel of window.

C5. $|m-s| > t$

where t is a threshold. Distinguishes bars from constant intensity.

2.
$$f_0 = s$$
$$f_1 = (l-1)s + (2-l)m$$
$$f_2 = m$$
$$f_3 = (r-3)m + (4-r)s$$
$$f_4 = s$$

Design of bar detector.

3. s: $$(a_0 - s) + [a_1 - (l-1)s - (2-l)m](l-1) + [a_3 - (r-3)m - (4-r)s](4-r) + (a_4 - s) = 0$$

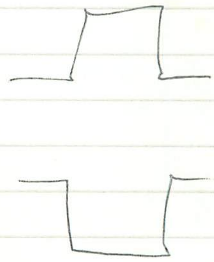
m: $$[a_1 - (l-1)s - (2-l)m](2-l) + (a_2 - m) + [a_3 - (r-3)m - (4-r)s](r-3) = 0$$

l: $$[a_1 - (l-1)s - (2-l)m](s-m) = 0$$

r: $$[a_3 - (r-3)m - (4-r)s](m-s) = 0$$

assuming $s \neq m$, we get:

$$s = \tfrac{1}{2}(a_0 + a_4)$$
$$m = a_2$$
$$l = 1 + \frac{a_2 - a_1}{a_2 - \frac{1}{2}(a_0 + a_4)}$$
$$r = 4 - \frac{a_2 - a_3}{a_2 - \frac{1}{2}(a_0 + a_4)}$$



4. c1. $$1 + \frac{a_2 - a_1}{a_2 - \frac{1}{2}(a_0 + a_4)} > 1$$

i.e. if $a_2 > \frac{1}{2}(a_0 + a_4)$
then $\cancel{a_1 < \frac{1}{2}(a_0 + a_4)}$ $a_2 > a_1$
else $\cancel{a_1 > \frac{1}{2}(a_0 + a_4)}$ $a_2 < a_1$

c2. $$1 + \frac{a_2 - a_1}{a_2 - \frac{1}{2}(a_0 + a_4)} \leqslant 2$$

i.e. if $a_2 > \frac{1}{2}(a_0 + a_4)$
then $a_1 \geqslant \frac{1}{2}(a_0 + a_4)$
else $a_1 \leqslant \frac{1}{2}(a_0 + a_4)$

c3. $$4 - \frac{a_2 - a_3}{a_2 - \frac{1}{2}(a_0 + a_4)} > 3$$

i.e. if $a_2 > \frac{1}{2}(a_0 + a_4)$
then $a_3 > \frac{1}{2}(a_0 + a_4)$
else $a_3 < \frac{1}{2}(a_0 + a_4)$

c4. $$4 - \frac{a_2 - a_3}{a_2 - \frac{1}{2}(a_0 + a_4)} \leqslant 4$$

i.e. if $a_2 > \frac{1}{2}(a_0 + a_4)$
then $a_3 \leqslant a_2$
else $a_3 \geqslant a_2$

c5. $$\left| a_2 - \tfrac{1}{2}(a_0 + a_4) \right| > t$$

Design of bar detector.

5. $$n_0 = a_0 - s = \tfrac{1}{2}(a_0 - a_4)$$

$$n_1 = a_1 - (l-1)s - (2-l)m$$

$$= a_1 - \frac{(a_2 - a_1) \cdot \frac{1}{2}(a_0 + a_4)}{a_2 - \frac{1}{2}(a_0 + a_4)} - \left(1 - \frac{a_2 - a_1}{a_2 - \frac{1}{2}(a_0 + a_4)}\right) a_2$$

$$= 0$$

$$n_2 = a_2 - m = 0$$

$$n_3 = a_3 - (r-3)m \mp (4-r)s = 0$$

$$n_4 = a_4 - s = \cancel{\tfrac{1}{2}(a_0 - a_4)}\ \tfrac{1}{2}(a_4 - a_0)$$

There is only one test needed: $\frac{1}{2}|a_0 - a_4| < n$, where n is a threshold.

Consider a bar with $m > s$. By adding a noise component of $(m-s)$ to f_0 we can make the bar look like an edge. So, to distinguish bars from edges, we let $n = \frac{1}{2}|m-s|$
The test is then:

$$\cancel{4}\,|a_0 - a_4| < |a_2 - \tfrac{1}{2}(a_0 + a_4)|$$